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METHOD OF DESCRIPTIVE REGULARIZATION AND QUALITY OF APPROXIMATE SOLUTIONS

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A method of solving Fredholm integral equations of the first kind is described, which is based on the a priori knowledge of the arrangement of extrema and inflection points of the desired solution and permits taking account of the fundamental qualitative regularities inherent in the exact solution of the problem.

1°. The mathematical theory of the solution of incorrectly formulated problems has been developed sufficiently well at this time [1, 2]. The main point of this theory is the use of a priori information about the accuracy of giving the entrance data and (or) about the desired solution to some extent. The nature of this information can be twofold: quantitative or qualitative. As a rule, the majority of methods use quantitative information about the accuracy of giving the entrance data and quite general information about the "smoothness" of the solution (the Tikhonov regularization method, the residual method). The distinctive peculiarity of the Ivanov method of quasisolutions is the possibility of using not only information of the type mentioned, but also just qualitative information associated with the a priori representations of the behavior of the desired solution. As a rule, an objective basis for the presence of such information is intuitive considerations about the simplicity of the structure of the desired solution as well as certain general conceptions about the behavior of the physical process being studied. The former are related to the natural tendency of the researcher to identify the most important and essential items in the mathematical model and can also be dictated by fully defined esthetic considerations.

The latter appear, for example, when a perfectly evident fact in the study of the brightness distribution of a star is the drop in intensity from the center of the star to its edges if, certainly, the star is unitary, and the presence of two maxima if the star is binary.

Let us assume that the phenomenon being studied is characterized quantitatively by the function $u = u(x)$, $a \leq x \leq b$. Such quantitative characteristics as the variation in the function $u(x)$, the root-mean-square value of its k-th derivative, etc., which are often used in solving incorrect problems, can be taken as a measure of its "simplicity." It is also well known that the behavior of a function is modeled sufficiently effectively on an intuitive level if the possible arrangement of its characteristic points, extremum points, and the change in curvature is given. It is hence considered that, on the whole, the function will behave in a natural manner, i.e., is single-valued, has no reentrant points, is sufficiently smooth, and therefore, can be drawn with one "stroke" of the pen. Such a class of simple functions can be given if sections of their monotonicity and convexity are indicated. The class of smooth functions with $L - 1$ sections of monotonicity can be written by the conditions

$$M = \{u(x) : (-1)^{l+i} u'(x) \leqslant 0,\ x_i \leqslant x \leqslant x_{i+1},\ i = 1,2, \ldots, L-1\},$$

where x_i, $i = 1, 2, \ldots, L$ are extrema of the function $u(x)$, $a = x_1 < x_2 < \ldots < x_L = b$ and the parameter l, equal to 1 or 2, governs the nature of the monotonicity in the first section. It is hence assumed that $M = M(x_2, \ldots, x_{L-1}; l, L)$, i.e., the number of extrema, the alternation of sections of growth and decrease in the function, and also the arrangement of the inner extrema can vary. Great detail in the class of functions being considered will be achieved if sections with curvature of constant sign are also extracted. We then arrive at the class

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$$V = \{u(x) : (-1)^{v+i}u''(x) \leqslant 0,\ \bar{x}_j \leqslant x \leqslant \bar{x}_{j+1},\ j = 1,2,\ldots,\ N-1\},$$

where $\bar{x}_j$, j = 2, 3, ..., N − 1 are inflection points of the function u(x); $a = \bar{x}_1 < \bar{x}_2 < \ldots < \bar{x}_N = b$; the parameter v, equal to 1 or 2, governs the sign of the curvature in the first section; and N − 1 is the number of curvature sections of constant sign. It is assumed that $V = V(\bar{x}_2, \ldots, \bar{x}_{N-1}; v, N)$.

The nodes x_i and $\bar{x}_j$ should satisfy the following matching conditions. If $x_1 < \bar{x}_2 < x_2$, then it is necessary that $x_2 < \bar{x}_3 < x_3, \ldots, x_{L-2} < \bar{x}_{L-1} < x_{L-1}$, but a change in the curvature can be in the last section $[x_{L-1}, b]$ (and then $x_{L-1} < \bar{x}_L < b$ and N = L + 1) or cannot be (and then $\bar{x}_L = b$ and N = L). If $x_2 < \bar{x}_2 < x_3$, i.e., the curvature is constant in the section $[x_1, x_2]$, then it is necessary that $x_3 < \bar{x}_3 < x_4, \ldots, \bar{x}_{L-1} < x_L$, and $\bar{x}_{L-1} = x_L$, i.e., N = L − 1, in the last section in the case of no change in curvature, or $x_{L-1} < \bar{x}_{L-1} < x_L$ in the case of a change and then $\bar{x}_L = b$ and N = L.

These conditions correspond to intuitive considerations about the "simplicity" of the behavior of a function from the class under consideration and are realized in a significant number of applied problems.

Methods of solving Fredholm integral equations of the first kind

$$Ku \equiv \int_a^b k(x, y)\, u(y)\, dy = f(x), \quad c \leqslant x \leqslant d,$$

where $f(x) \in L_2[c, d]$, the kernel $k(x, y) \in L_2(\Omega)$, $\Omega = [c, d] \times [a, b]$ with the implication of the conditions M and (or) V we will call descriptive methods of regularization [3]. Namely, we consider the following mathematical programming problem: Find the function u*(y) which is an approximate solution of the problem

$$\|Ku - f\|_{L_2} \to \min_{u \in D},$$

where D = M, V or M∩V. The kernel and right side of the equation under consideration can hence be given approximately.

A singularity of the descriptive regularization method is the description of an admissible set of functions by the extraction of characteristic points (extremal points and the change in sign of the curvature) and the assumption of "simplicity" of the structure of the desired solution in the sense mentioned above.

It is known [4] that the conditions M stabilize approximate solutions in a uniform metric. As Samarin has recently shown, the implication of convexity conditions assures stabilization of the first order of smoothness, i.e., uniform convergence of the approximations together with the derivatives holds under definite conditions.

It is easy to see that numerical discretization of the problem results in a quadratic programming problem with specific linear constraints. In this paper the principal attention is paid to the construction of effective numerical algorithms for the solution of the discretized problem with its specifics taken into account, and their "operation" verified in a series of model problems. We are hence also interested in the quality of the approximation, defined by their character itself. Let us note that the quality of the approximations is determined not so much by their accuracy as by the appearance of fundamental regularities inherent in the exact solution, and depends greatly on the esthetic perception of these approximations.

2°. Let us go over to a discrete formulation of the problem. Let us introduce the mesh of nodes $\omega_x^h = \{x_i,\ i = \overline{1, m}\}$ on a segment [c, d] and $\omega_y^h = \{y_i,\ j = \overline{1, n}\}$ on a segment [a, b]. Assuming $u(y_j) = u_j$, we can write

$$Ku \approx \sum_{j=1}^{n} q_j k(x, y_j)\, u_j = f(x),$$

where q_j are quadrature coefficients (the trapezoid formula later) and setting $x = x_i$ here, we arrive at the following collocation conditions:

$$A\hat{u} = \hat{f}, \tag{1}$$

where A is a matrix of order m × n with elements $a_{ij} = q_j k(x_i, y_j)$, the vector $\hat{u} = (u_1, u_2, \ldots, u_n)$, and the vector $\hat{f} = (f_1, f_2, \ldots, f_m)$, $f_i = f = f(x_i)$. We select

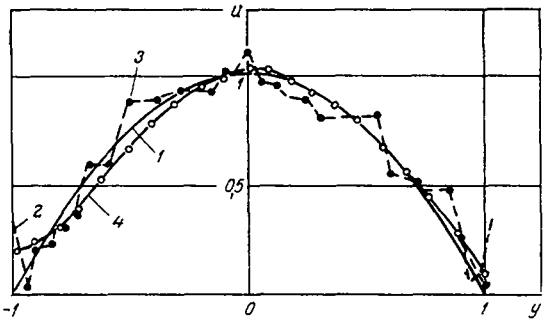

Fig. 1. Results of a numerical computation of the first model problem with the errors k – 5%, f – 3%; u_{exact}; 2) u_{approx} for ρ_i = 0, i = $\overline{1.40}$ without taking account of constraints; 3) u_{approx} for ρ_i = 0, i = $\overline{1.40}$ with the conditions M taken into account; 4) u_{approx} for ρ_i = 1, i = $\overline{1.40}$ taking account of the conditions M.

$$\Phi[\hat{u}] = \Phi[\hat{u};\ \hat{f}] = \sum_{i=1}^{m} p_i \left(\sum_{j=1}^{n} a_{ij}u_j - f_i\right)^2 = \|A\hat{u} - \hat{f}\|_p^2,$$

as the target functional, where $\rho_i > 0$ are certain weight coefficients.

Writing the monotonicity and convexity conditions for discrete functions $\hat{u}$ and retaining the previous notation for them, we arrive at the necessity to solve the following mathematical programming problem:

$$\Phi[u] \to \min_{u \in D} \tag{2}$$

with specific constraints of the linear inequality type.

We chose the method of the projection of conjugate gradients as the numerical algorithm to solve the problem (2). We were hence guided by the known fact that this method, among the quasi-Newtonian ones, results in a solution for quadratic functionals without constraints in a finite number (not more than n) steps, equal to the rank of the matrix A. In this case we apparently have a situation which does not agree with but is close to that when the rank of the problem is in practice substantially less than n, i.e., is quite small.

The following reasoning is the foundation for this. Every regularization method practically results in a reduction in the influence of the "high-frequency" components of the solution in some special basis, which is equivalent to replacing the initial problem by one similar but already degenerate problem with a low rank, and it is known from practice that the solution of degenerate problems is realized numerically more effectively than the solution of the so-called poorly specified problems of high dimensionality. The numerical experiments presented below agree well with these considerations since the actual number of iterations is not very large.

Another consideration which guided us is the diminution in the calculation time and the rise in accuracy of the calculations because of the effective realization of the projection operation itself. We succeeded in reaching such a target for sets D of M or V type by applying an algorithm developed for these cases by one of the authors [5, 6], and for the set $D = M \cap V$ by successive utilization of additional information [3].

Let us present the computational formulas for the method of projection of conjugate gradients for the problem (2).

Let $u^0 \in D$ be the initial approximation. The iteration process is constructed by means of the formulas

$$\begin{gathered} u^{s+1} = \mathcal{P}_D(u^s - \alpha_s g^s),\ s = 0, 1, 2, \ldots, \\ g^0 = \operatorname{grad}\Phi[u^0],\ g^s = \operatorname{grad}\Phi[u^s] - \beta_s g^{s-1},\ s = 1, 2, \ldots, \\ \beta_s = \frac{(\operatorname{grad}\Phi[u^s],\ \operatorname{grad}\Phi[u^{s-1}] - \operatorname{grad}\Phi[u^s])}{\|\operatorname{grad}\Phi[u^{s-1}]\|^2}. \end{gathered} \tag{3}$$

Here $\mathcal{P}_D$ is the projection operation on the set D

$$\|\mathcal{P}_D(z) - z\| = \min_{w \in D}\|w - z\|.$$

The magnitude of the descent step α_s is selected from the condition of a monotonic decrease in the target functional

$$\Phi[u^s - \alpha_s g^s] \leqslant \Phi[u^s]. \tag{4}$$

The value

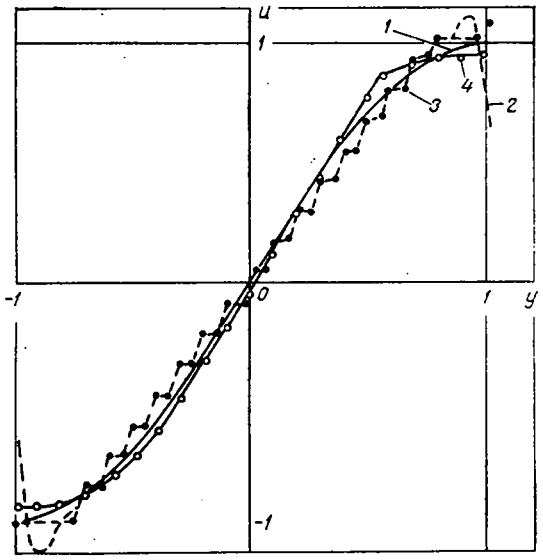

Fig. 2. Results of a numerical computation of the second model problem with the errors $k - 3\%, f - 2\%$: 1) u_{exact}; 2) u_{approx} for $\rho_i = 0$, $i = \overline{1.40}$ without taking account of constraints; 3) u_{approx} for $\rho_i = 0$, $i = \overline{1.40}$ with the conditions M taken into account; 4) u_{approx} for $\rho_i = 1.5$, $i = \overline{1.38}$, $\rho_i = 0.5$, $i = 39.40$ with the conditions M taken into account.

$$\alpha_s^* = \frac{(\operatorname{grad}\Phi[u^s],\ g^s)}{2(A'PAg^s, g^s)},$$

$$\Phi[u^s - \alpha_s^* g^s] = \min_{\alpha \geq 0} \Phi[u^s - \alpha g^s],$$

can be used to determined α_s by setting $\alpha_s = \alpha_s^*$ in the case of compliance with condition (4) and setting $\alpha_s = \alpha_s^*/2^k$ otherwise ($k = 1, 2, \ldots$).

The gradient of the target functional in the construction of the iteration process (3) by the conjugate gradients method in a finite-dimensional analog of the space L_2 is

$$\operatorname{grad}\Phi[u] = 2(A'PAu - A'Pf), \tag{5}$$

where the prime denotes the transpose, and the matrix is $P = \operatorname{diag}(p_1, p_2, \ldots, p_m)$.

If it is known a priori that the solution possesses first-order smoothness, for example, then an increase in the accuracy and a rise in quality can be attained by applying the conjugate gradients method in a finite-dimensional analog of the space W_2^1. The gradient of the functional $\Phi[u]$ is then

$$\operatorname{grad}\Phi[u] = (\bar{\Phi}_1, \bar{\Phi}_2, \ldots, \bar{\Phi}_n),$$

where $\bar{\Phi}_j$ ($j = \overline{1, n}$) are determined from the system of linear equations

$$\frac{\rho_j}{\gamma_j}\bar{\Phi}_{j+1} - \left(1 + \frac{\rho_j}{\gamma_j} + \frac{\rho_{j-1}}{\gamma_j}\right)\bar{\Phi}_j + \frac{\rho_{j-1}}{\gamma_j}\bar{\Phi}_{j-1} = -\Phi_j,\ j = \overline{1,n},$$

$$\bar{\Phi}_0 = \bar{\Phi}_{n+1} = 0,\ \rho_0 = \rho_n = 0,$$

which can be solved by the factorization method. Here Φ_j ($j = \overline{1, n}$) are the components of the vector (5), $\gamma_j > 0$ and $\rho_j \geq 0$ ($j = \overline{1, n}$) are given numbers related to a specific normalization of W_2^1.

3°. To verify the efficiency of the method being proposed, a program was compiled in FORTRAN for the BÉSM-6 computer and numerical experiments to solve model problems with a known exact solution were performed.

The function $u_{exact}(y) = 1 - y^2$ with

$$k(x, y) = 1/(1 + (x - y)^2),$$

$$f(x) = (2 - x^2)(\operatorname{arctg}(1 - x) + \operatorname{arctg}(1 + x)) - 2 - x \ln\frac{1 + (1 - x)^2}{1 + (1 + x)^2},$$

was taken as the exact solution in the class of piecewise-monotonic functions for the first model problem and $u_{exact}(y) = \sin \pi/2\, y$ with $k(x, y) = (x - y)^2$, $f(x) = -16x/\pi^2$ was taken for the second model problem. The

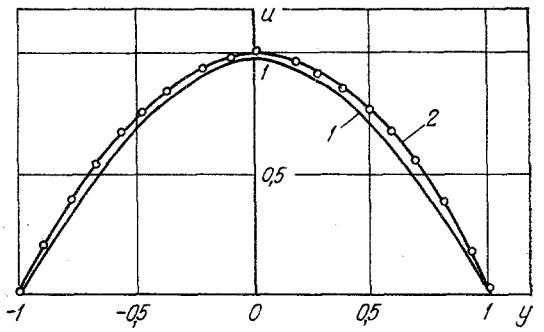

Fig. 3. Results of a numerical computation of the third model problem: 1) u_{exact}; 2) u_{approx} with $\rho_i = 0$, $i = \overline{1,40}$ taking account of the conditions V.

computations were executed on uniform meshes $\omega_x^h \in [-2.2]$, $\omega_y^h \in [-1.1]$ with the steps $h_x = 0.1$, $h_y = 0.05$, respectively. The function $u^0 = 0$ was taken as the initial approximation. An explicit form of the projector on the set of piecewise-monotonic functions described in [6] was used in the projection on M. To verify the stability of the method relative to errors in the entrance data, computations were performed with uniformly distributed random perturbations of the kernel k and the right side f.

As follows from the results of computations presented in Figs. 1 and 2, taking account of the constraints M as well as the selection of the metric of the space (specifically, the parameters ρ_j, $j = \overline{1, n-1}$) permits obtaining an approximate solution sufficiently close to the exact solution.

The function $u_{exact}(y) = \cos[(\pi/2)y]$ with $k(x, y) = x - y$, $f(x) = 4x/\pi$, $x \in [-2.2]$, $y \in [-1.1]$, $h_x = 0.1$, $h_y = 0.05$ was taken as the exact solution for the third model problem in the class of convex functions. The solution of this problem is represented in Fig. 3.

Roughly, ten iterations are required for agreement of values of the functional $\Phi[u]$ with 10^{-7} accuracy at two adjacent iterations in each of the three problems, which corresponds approximately to 1 min of BÉSM-6 computer machine time (together with program translation). Cutting down the machine time should be achieved by taking the result obtained after a small number of iterations by the method of conjugate gradients without taking account of any constraints, as the initial approximation in the method of descriptive regularization.

A direct comparison of Figs. 1, 2, and Fig. 3 shows that taking account of just conditions of type M is not sufficient for qualification of the approximations obtained as completely qualitative although the accuracy of the approximation is totally satisfactory. The approximations in Figs. 1 and 2 are "stepwise" in nature and agree completely with the deductions obtained in [6]. At the same time, taking account of the conditions V results in a qualitative reproduction of the desired exact solution (Fig. 3). This deduction is probably not related to the particular examples considered, but is sufficiently general in nature.

Let us note that improvement in the perception of the approximations is also achieved in examining the conditions M when going over to descent according to the conjugate gradient in some other space which takes account of the smoothness of the function desired. The same effect can be achieved if it is taken into account that the descriptive regularization method is, as is the method of quasisolutions generally [7], a limit case of the Tikhonov method of regularization

$$\|Ku - f\|_{L_2}^2 + \alpha \|u^{(k)}\|_{L_2}^2 \to \min_{u \in D},$$

when the regularization parameter $\alpha > 0$ is sufficiently small but retains its influence as a stabilizing factor.

An analysis of the calculation expenditures, the electronic computer memory, and time needed to solve the problem of the descriptive regularization method shows that they are commensurate with the calculation expenditures needed to minimize finite-dimensional quadratic problems without constraints by the conjugate gradient method, and therefore, the descriptive regularization method is completely suitable for utilization in calculation practice.

The application of the descriptive regularization method is especially effective in the case when the operator K is single; i.e., the approximation problem is solved in the set of piecewise-monotonic functions.

The fundamental hypotheses and deductions of the research are carried over, without change, to nonlinear integral equations also, but as is seen from the above, explicit assignment of the operator is not absolute. This permits recommending the descriptive regularization method for a broad circle of inverse problems when the desired function is a function of one variable. The method of descriptive regularization can be

used effectively in the mode of the method of trials when the electronic computer has a display. A new kind of "resolver," whose main elements are an electronic computer performing the most routine part of the work, a display which permits operational analysis and decision making, and an operator-calculator which forms a new model for approbation on the basis of the data obtained, hence originates.

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SOLUTION OF INVERSE HEAT-CONDUCTION PROBLEMS ON SPECIALIZED ANALOG COMPUTERS

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Recommendations on the application of specialized analog computers for the solution of inverse problems of heat conduction are given. The presence of a zone of sensitivity delimiting the possible location of a primary information source is established.

Inverse problems are quite extensive in heat- and mass-transfer processes. This is explained primarily by the fact that measurement of the parameters of these processes (temperature, heat-flux density, etc., for instance) in the range of high values under non-steady-state conditions is difficult, and a completely insurmountable problem in a number of cases. In such situations inverse problems are the most acceptable method of solving these problems.

Inverse problems of heat conduction are used in thermal power plants to establish the thermal gasdynamic circumstances according to the results of temperature measurements, to determine uniqueness conditions, and for machine design. In connection with the growing heat loads, the determination of the thermal environment in the high-temperature range, i.e., the heat-flux density q_s and the surface temperature T_s, the temperature of the gas stream T_g flowing around a solid, the coefficient of heat transfer between the hot gas stream and the solid α_g, etc., according to the results of temperature measurements in the low-temperature range, is a problem which must be solved in engineering. Inverse problems of heat conduction are important in the design and construction of heat shields, in the prediction of the thermophysical properties of materials with a given operating range, etc.

If the process of heat transfer between a medium and a solid is considered, then depending on the location of the quantity to be determined inverse problems of heat conduction can be separated into three classes: internal, external, and combined. We shall refer such problems for which the parameters (characteristics) within the body or on its surfaces are determined as a result of the solution to internal, problems when the characteristics of the environment are found to external, and problems for which combinations of parameters of the first two classes will be the subject of solution to the combined classes. A diagram of the classification of inverse problems of heat conduction is shown in Fig. 1.

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